MATH 2060B: Mathematical Analysis II
Appendix: Compacts sets in $\mathbb{R}$

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1 Compact Sets in $\mathbb{R}$

Throughout this section, let (x_n) be a sequence in $\mathbb{R}$. Recall that a subsequence $(x_{n_k})_{k=1}^{\infty}$ of (x_n) means that $(n_k)_{k=1}^{\infty}$ is a sequence of positive integers satisfying $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$, that is, such sequence (n_k) can be viewed as a strictly increasing function $\mathbf{n} : k \in \{1, 2, ..\} \mapsto n_k \in \{1, 2, ...\}$.
In this case, note that for each positive integer N, there is $K \in \mathbb{N}$ such that $n_K \geq N$ and thus we have $n_k \geq N$ for all $k \geq K$.

Let us first recall the following two important theorems in real line.

Theorem 1.1 Nested Intervals Theorem *Let $(I_n := [a_n, b_n])$ be a sequence of closed and bounded intervals. Suppose that it satisfies the following conditions.*

(i) : $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$.

(ii) : $\lim_n (b_n - a_n) = 0$.

Then there is a unique real number ξ such that $\bigcap_{n=1}^{\infty} I_n = \{\xi\}$.

Proof: See [1, Theorem 2.5.2, Theorem 2.5.3]. □

Theorem 1.2 (Bolzano-Weierstrass Theorem) *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof: See [1, Theorem 3.4.8]. □

Definition 1.3 A subset A of $\mathbb{R}$ is said to be *compact* (more precise, *sequentially compact*) if every sequence in A has a convergent subsequence with the limit in A.

We are now going to characterize the compact subsets of $\mathbb{R}$. The following is an important notation in mathematics.

Definition 1.4 A subset A is said to be *closed* in $\mathbb{R}$ if it satisfies the condition:

if (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim x_n \in A$.

Example 1.5 (i) $\{a\}; [a, b]; [0, 1] \cup \{2\}$; $\mathbb{N}$; the empty set $\emptyset$ and $\mathbb{R}$ all are closed subsets of $\mathbb{R}$.

(ii) (a, b) and $\mathbb{Q}$ are not closed.

The following Proposition is one of the basic properties of a closed subset which can be directly shown by the definition. So, the proof is omitted here.

Proposition 1.6 *Let A be a subset of $\mathbb{R}$. The following statements are equivalent.*

(i) A is closed.

(ii) For each element $x \in \mathbb{R} \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$.

The following is an important characterization of a compact set in $\mathbb{R}$. **Warning:** this result is not true for the so-called *metric spaces* in general.

Theorem 1.7 *Let A be a closed subset of $\mathbb{R}$. Then the following statements are equivalent.*

(i) A is compact.

(ii) A is closed and bounded.

Proof: It is clear that the result follows if $A = \emptyset$. So, we assume that A is non-empty.
For showing $(i) \Rightarrow (ii)$, assume that A is compact.
We first claim that A is closed. Let (x_n) be a sequence in A. Then by the compactness of A, there is a convergent subsequence (x_{n_k}) of (x_n) with $\lim_k x_{n_k} \in A$. So, if (x_n) is convergent, then $\lim_n x_n = \lim_k x_{n_k} \in A$. Therefore, A is closed.
Next, we are going to show the boundedness of A. Suppose that A is not bounded. Fix an element $x_1 \in A$. Since A is not bounded, we can find an element $x_2 \in A$ such that $|x_2 - x_1| > 1$. Similarly, there is an element $x_3 \in A$ such that $|x_3 - x_k| > 1$ for $k = 1, 2$. To repeat the same step, we can obtain a sequence (x_n) in A such that $|x_n - x_m| > 1$ for $m \neq n$. From this, we see that the sequence (x_n) does not have a convergent subsequence. In fact, if (x_n) has a convergent subsequence (x_{n_k}). Put $L := \lim_k x_{n_k}$. Then we can find a pair of sufficient large positive integers p and q with $p \neq q$ such that $|x_{n_p} - L| < 1/2$ and $|x_{n_q} - L| < 1/2$. This implies that $|x_{n_p} - x_{n_q}| < 1$. It leads to a contradiction because $|x_{n_p} - x_{n_q}| > 1$ by the choice of the sequence (x_n). Thus, A is bounded.
It remains to show $(ii) \Rightarrow (i)$. Suppose that A is closed and bounded.
Let (x_n) be a sequence in A. Thus, (x_n). Then the Bolzano-Weierstrass Theorem assures that there is a convergent subsequence (x_{n_k}). Then by the closeness of A, $\lim_k x_{n_k} \in A$. Thus A is compact.
The proof is finished.
□

For convenience, we call a collection of open intervals $\{J_\alpha : \alpha \in \Lambda\}$ an *open intervals cover* of a given subset A of $\mathbb{R}$, where Λ is an arbitrary non-empty index set, if each J_α is an open interval (not necessary bounded) and

$$A \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha.$$

Theorem 1.8 Heine-Borel Theorem: *Any closed and bounded interval $[a,b]$ satisfies the following condition:*

(HB) *Given any open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of $[a,b]$, we can find finitely many $J_{\alpha_1},..,J_{\alpha_N}$ such that $[a,b]\subseteq J_{\alpha_1}\cup\cdots\cup J_{\alpha_N}$*

Proof: Suppose that $[a,b]$ does not satisfy the above Condition (HB). Then there is an open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of $[a,b]$ but it it has no finite sub-cover. Let $I_1 := [a_1,b_1] = [a,b]$ and m_1 the mid-point of $[a_1,b_1]$. Then by the assumption, $[a_1,m_1]$ or $[m_1,b_1]$ cannot be covered by finitely many J_α's. We may assume that $[a_1,m_1]$ cannot be covered by finitely many J_α's. Put $I_2 := [a_2,b_2] = [a_1,m_1]$. To repeat the same steps, we can obtain a sequence of closed and bounded intervals $I_n = [a_n,b_n]$ with the following properties:

(a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots$;

(b) $\lim_n (b_n - a_n) = 0$;

(c) each I_n cannot be covered by finitely many J_α's.

Then by the Nested Intervals Theorem, there is an element $\xi \in \bigcap_n I_n$ such that $\lim_n a_n = \lim_n b_n = \xi$. In particular, we have $a = a_1 \le \xi \le b_1 = b$. So, there is $\alpha_0 \in \Lambda$ such that $\xi \in J_{\alpha_0}$. Since J_{α_0} is open, there is $\varepsilon > 0$ such that $(\xi-\varepsilon,\xi+\varepsilon)\subseteq J_{\alpha_0}$. On the other hand, there is $N\in\mathbb{N}$ such that a_N and b_N in $(\xi-\varepsilon,\xi+\varepsilon)$ because $\lim_n a_n = \lim_n b_n = \xi$. Thus we have $I_N = [a_N,b_N]\subseteq(\xi-\varepsilon,\xi+\varepsilon)\subseteq J_{\alpha_0}$. It contradicts to the Property (c) above. The proof is finished.
□

Remark 1.9 The assumption of the closeness and boundedness of an interval in Heine-Borel Theorem is essential.
For example, notice that $\{J_n := (1/n,1) : n = 1,2...\}$ is an open interval covers of $(0,1)$ but you cannot find finitely many J_n's to cover the open interval $(0,1)$.

The following is a very important feature of a compact set.

Theorem 1.10 *Let A be a subset of $\mathbb{R}$. Then the following statements are equivalent.*

(i) **Heine-Borel property:** *For any open intervals cover $\{J_\alpha\}_{\alpha\in\Lambda}$ of A, we can find finitely many $J_{\alpha_1},..,J_{\alpha_N}$ such that $A\subseteq J_{\alpha_1}\cup\cdots\cup J_{\alpha_N}$.*

(ii) A is compact.

(iii) A is closed and bounded.

Proof: The result will be shown by the following path

$$(i)\Rightarrow(ii)\Rightarrow(iii)\Rightarrow(i).$$

For $(i)\Rightarrow(ii)$, assume that the condition (i) holds but A is not compact. Then there is a sequence (x_n) in A such that (x_n) has no subsequent which has the limit in A. Put $X =$

$\{x_n : n = 1, 2, ...\}$. Then X is infinite. Also, for each element $a \in A$, there is $\delta_a > 0$ such that $J_a := (a-\delta_a, a+\delta_a) \cap X$ is finite. Indeed, if there is an element $a \in A$ such that $(a-\delta, a+\delta) \cap A$ is infinite for all $\delta > 0$, then (x_n) has a convergent subsequence with the limit a. On the other hand, we have $A \subseteq \bigcup_{a \in A} J_a$. Then by the compactness of A, we can find finitely many $a_1, ..., a_N$ such that $A \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. So we have $X \subseteq J_{a_1} \cup \cdots \cup J_{a_N}$. Then by the choice of J_a's, X must be finite. This leads to a contradiction. Therefore, A must be compact.
The implication $(ii) \Rightarrow (iii)$ follows from Theorem 1.7 at once.
It remains to show $(iii) \Rightarrow (i)$. Suppose that A is closed and bounded. Then we can find a closed and bounded interval $[a, b]$ such that $A \subseteq [a, b]$. Now let $\{J_\alpha\}_{\alpha \in \Lambda}$ be an open intervals cover of A. Notice that for each element $x \in [a, b] \setminus A$, there is $\delta_x > 0$ such that $(x - \delta_x, x + \delta_x) \cap A = \emptyset$ since A is closed by using Proposition 2.4. If we put $I_x = (x - \delta_x, x + \delta_x)$ for $x \in [a, b] \setminus A$, then we have

$$[a, b] \subseteq \bigcup_{\alpha \in \Lambda} J_\alpha \cup \bigcup_{x \in [a,b] \setminus A} I_x.$$

Using the Heine-Borel Theorem 1.8, we can find finitely many J_α's and I_x's, say $J_{\alpha_1}, ..., J_{\alpha_N}$ and $I_{x_1}, ..., I_{x_K}$, such that $A \subseteq [a, b] \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N} \cup I_{x_1} \cup \cdots \cup I_{x_K}$. Note that $I_x \cap A = \emptyset$ for each $x \in [a, b] \setminus A$ by the choice of I_x. Therefore, we have $A \subseteq J_{\alpha_1} \cup \cdots \cup J_{\alpha_N}$ and hence A is compact.
The proof is finished. □

Remark 1.11 In fact, the condition in Theorem 1.10(i) is the usual definition of a *compact set* for a general topological space. More precise, if a set A satisfies the Definition 1.4, then A is said to be *sequentially compact.* Theorem 1.10 tells us that the notation of the compactness and the sequentially compactness are the same as in the case of a subset of $\mathbb{R}$. However, these two notation are different for a general topological space.

Strongly recommended: take the courses: MATH 3060; MATH3070 for the next step.

2 Appendix: Open subsets of $\mathbb{R}$

Definition 2.1 Let V be a subset of $\mathbb{R}$.

(i) A point $c \in V$ is called an interior point of V if there is $r > 0$ such that $(c-r, c+r) \subseteq V$.

(ii) V is said to be an open subset of $\mathbb{R}$ is for every element in V is an interior point of V. In this case, if $x_0 \in V$, then V is called an open neighborhood of the point x_0.

Example 2.2 With the notation as above, we have

(i) All open intervals are open subsets of $\mathbb{R}$.

(ii) $\emptyset$ and $\mathbb{R}$ are open subsets.

(iii) Any closed and bounded interval is not an open subset.

(iv) The set of all rational numbers $\mathbb{Q}$ is neither open nor closed subset.

Proposition 2.3 *A non-empty subset A of $\mathbb{R}$ is open if and only if there is sequence of open intervals $I_n = (a_n, b_n)$ for $n = 1, 2, ...$ such that $A = \bigcup_{n=1}^{\infty} I_n$ and $I_n \cap I_m = \emptyset$ for $m \neq n$.*

Proof: Assume that A is an open subset. Notice that $\overline{\mathbb{Q}} = \mathbb{R}$. Since A is open, we see that $A \cap \mathbb{Q}$ is also a non-empty countable subset. Let $A \cap \mathbb{Q} = \{x_1, x_2, ...\}$. For each x_k, put $I_k := \bigcup\{J : x_k \in J \text{ and } J \text{ is an open interval}\}$. Then $X = \bigcup_{k=1}^{\infty} I_k$. On the other hand, we notice that I_k is also any open interval **(Why??)**. From this, we see that $I_k \cap I_j = \emptyset$ or $I_k = I_j$. Thus, we can find a subsequence (x_{n_k}) such that $I_{n_k} \cap I_{n_j} = \emptyset$ for $k \neq j$. Thus the sequence of disjoint open intervals $(I_{n_k})_{k=1}^{\infty}$ that we want.
The converse is clear. □

Recall that a point $c \in \mathbb{R}$ is called a *limit point (or cluster point)* of a subset A of $\mathbb{R}$ if for any $\delta > 0$, we have $(c - \delta, c + \delta) \cap A \neq \emptyset$.
Moreover, A is said to be a closed subset of $\mathbb{R}$ if A contains all its limit points. Let us recall the following useful fact that we have used many times.

Proposition 2.4 *Let A be a subset of $\mathbb{R}$. Then the following statements are equivalent.*

(i) A is closed.

(ii) If (x_n) is a sequence in A and $\lim x_n$ exists, then $\lim x_n \in A$.

The following an important relation between the notion of openness and closeness.

Proposition 2.5 *A subset A of $\mathbb{R}$ is open if and only if its complement $A^c = \mathbb{R} \setminus A$ is closed in $\mathbb{R}$.*

Proof: For ($\Rightarrow$), we assume that A is open first. If A^c does not have the limit points, then the set A^c is clearly a closed set by the definition. Now let c be a limit point of A^c. □

References

[1] R.G. Bartle and I.D. Sherbert, Introduction to Real Analysis, (4*th ed*), Wiley, (2011).